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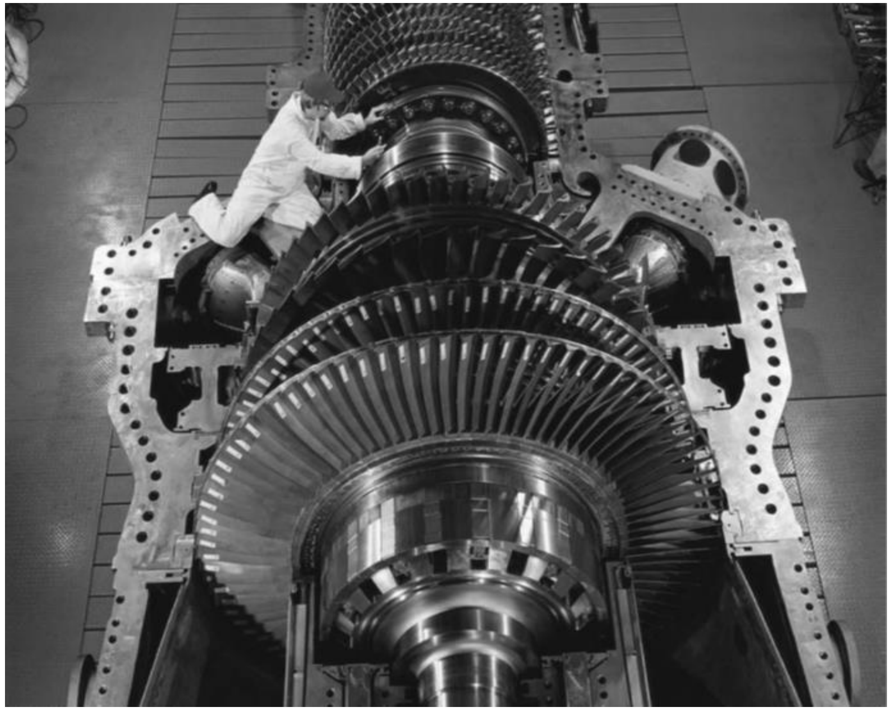

ME 113: Mechanics of Deformable Bodies

Torsion

Torsion of circular bar, Transmission of power by circular shafts

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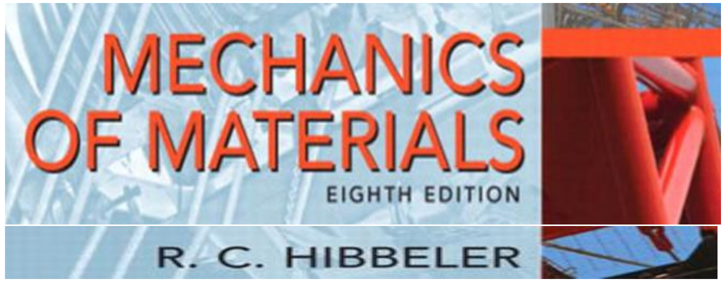
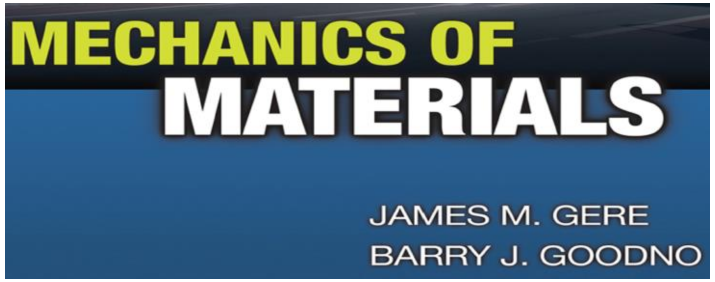
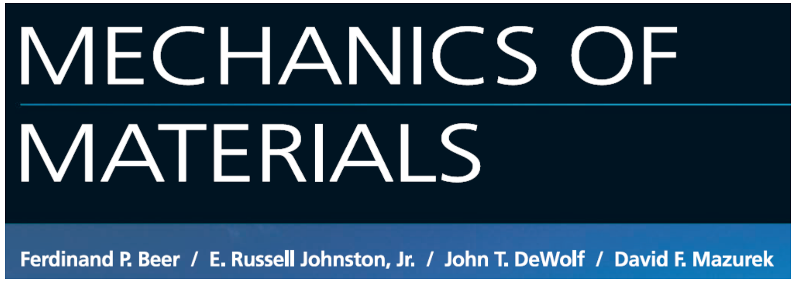
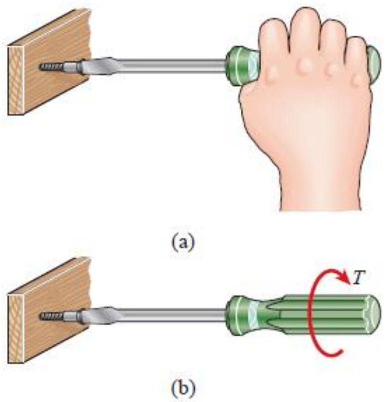
Adapted from:

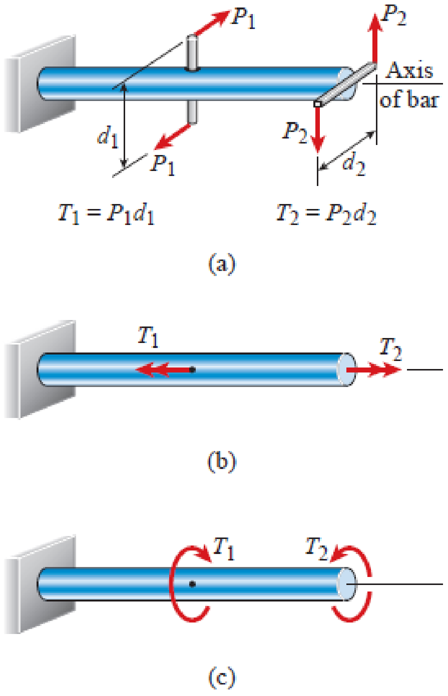






Torsion





Torsion refers to the twisting of a straight bar when it is loaded by moments (or torques) that tend to produce rotation about the longitudinal axis of the bar

Similar to turning a screwdriver hand which applies a torque T and twists the shank of the screwdriver.

An idealized case of torsional loading is by two pairs of equal and opposite forces. Each pair of forces forms a couple that tends to twist the bar about its longitudinal axis. The torque or twisting moment is the product of one of the forces and the perpendicular distance between the lines of action of the forces.

The moments are $\boldsymbol{T_1 = P_1 d_1}$ and the second is $\boldsymbol{T_2 = P_2 d_2}$

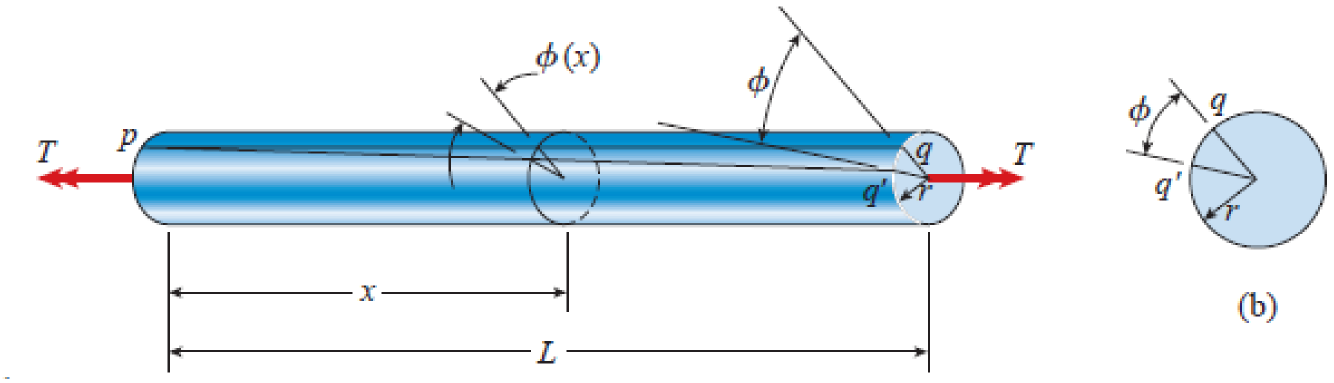
The SI unit for moment is the newton meter (Nm). The moment of a couple may be represented by a vector in the form of a double-headed arrow

Torsional Deformation

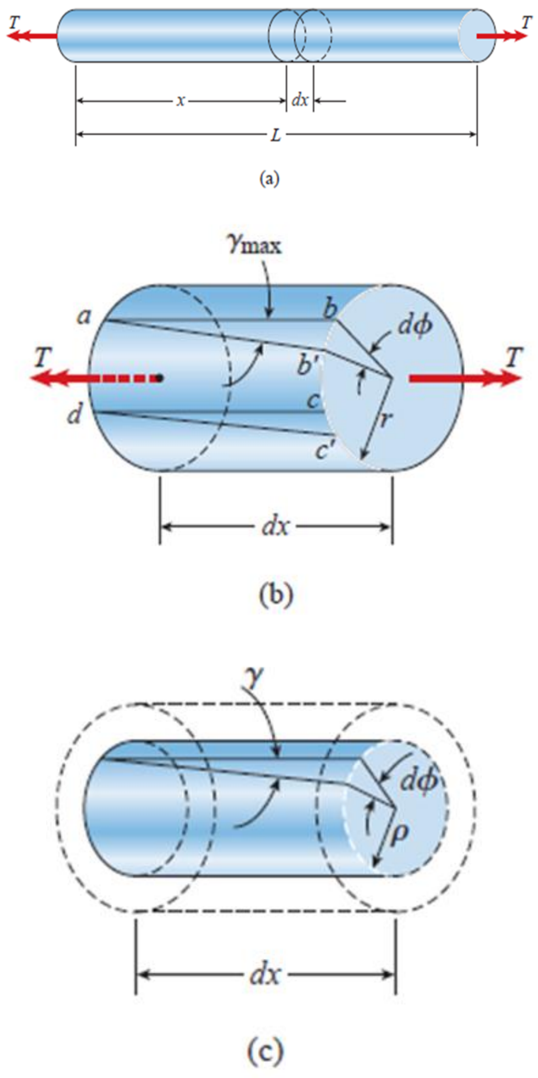
Let the left hand end of the bar is fixed in position with the action of the torque T. The right-hand end will rotate through a small angle **ϕ** known as the angle of twist (or angle of rotation).

Because of this rotation, a straight longitudinal line *pq* on the surface of the bar will become a helical curve *pq'*.

The angle of twist changes along the axis of the bar, and at intermediate cross sections it will have a value **ϕ**(x). If every cross section of the bar has the same radius and is subjected to the same torque (pure torsion), the angle **ϕ** (x) will vary linearly between the ends.



Shear Strain



The magnitude of the shear strain at the outer surface of the bar, denoted Υ_{max} is equal to the decrease in the angle at point a. The decrease in angle ***bad*** is ϒ as shown

$$\gamma_{max} = \frac{bb'}{ab}$$

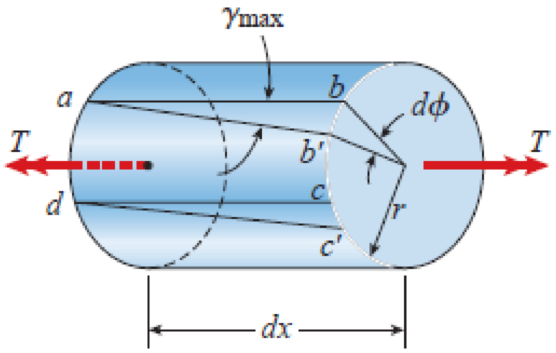
the distance bb' is also equal to (r dϕ), where d ϕ also is measured in radians. Thus, the preceding equation becomes

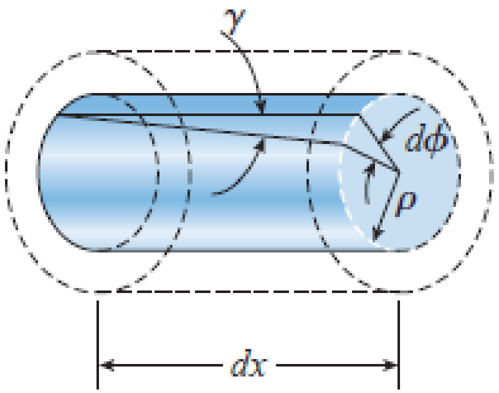
$$\gamma_{max} = \frac{r d\phi}{dx}$$

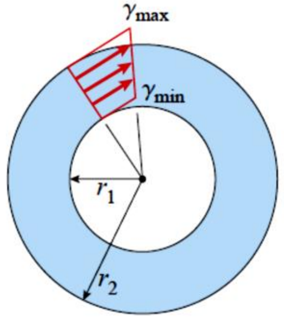
This relates the shear strain at the outer surface to the angle of twist. The quantity dϕ/dx is the rate of change of the angle of twist wrt the distance x. Denote dϕ/dx by θ and refer to as the angle of twist per unit length

$$\theta = \frac{d\phi}{dx}$$

Shear Strain







We can now write the equation for the shear strain at the outer surface at x = L as

$$\gamma_{max} = \frac{r\,d\phi}{dx} = r\theta \qquad \gamma_{max} = r\theta = \frac{r\phi}{L}$$

The shear strain within the bar at radius ρ is Υ. This is

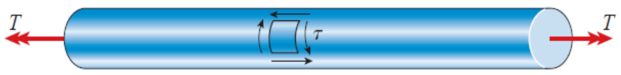
$$\gamma = \rho\theta = \frac{\rho}{r}\gamma_{max}$$

The linear variation in shear strain between the maximum strain at the outer surface and the minimum strain at the interior surface is

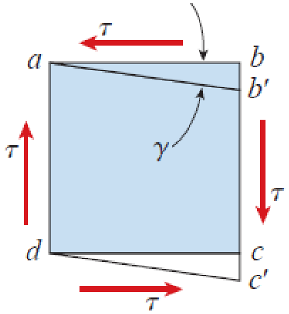
$$\gamma_{max} = \frac{r_2\phi}{L} \qquad \gamma_{min} = \frac{r_1}{r_2}\gamma_{max} = \frac{r_1\phi}{L}$$

r_1 and r_2 are the inner and outer radii, respectively, of the tube.

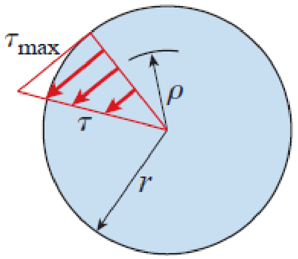
Shear Stress



(a)



(b)



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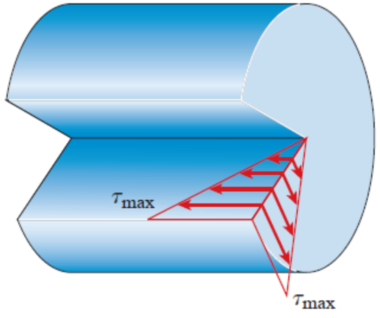


The magnitudes of the shear stresses can be determined from the strains by using the stress-strain relation for the material of the bar. For linearly elastic material, we can use Hooke's law in shear

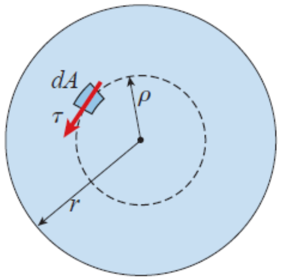
$$\tau = G\gamma$$

in which G is the shear modulus of elasticity and g is the shear strain in radians. Combining this equation with the equations for the shear strains

$$\tau_{max} = Gr\theta \qquad \tau = G\rho\theta = \frac{\rho}{r}\tau_{max}$$

in which τ_{max} is the shear stress at the outer surface of the bar (radius r), **τ** is the shear stress at an interior point (radius **ρ**)

The Torsion Formula



This provides the relationship between the shear stresses and the torque T. Consider an element of area dA located at radial distance r from the axis of the bar. The shear force acting on this element is equal to (τ dA), The moment of this force about the axis of the bar is equal to the force times its distance from the center, or (τ r dA). Substituting for the shear stress τ from Eq. we can express this elemental moment as

$$dM = \tau\rho dA = \frac{\tau_{\max}}{r}\rho^2 dA$$

The resultant moment (equal to the torque T) is the summation over the entire cross-sectional area of all such elemental moments:

$$T = \int_A dM = \frac{\tau_{\max}}{r}\int_A \rho^2 dA = \frac{\tau_{\max}}{r} I_P$$

in which $I_P = \int_A \rho^2 dA$

For a circle of radius r and diameter d, the polar moment of inertia is $I_P = \frac{\pi r^4}{2} = \frac{\pi d^4}{32}$

The Torsion Formula

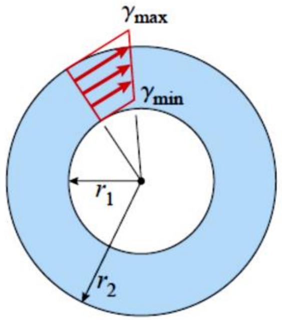
From the previous, $\frac{T}{I_P} = \frac{\tau}{r}$ and $\tau = Gr\theta$

Combining above, $\frac{T}{I_P} = \frac{\tau}{r} = \frac{G\phi}{L}$

Substituting r=d/2 and $I_P = \pi d^4/32$ into the torsion formula, we get the following equation for the maximum stress

$$\tau_{\max} = \frac{16T}{\pi d^3}$$

For tubes/hollow shafts



$$I_P = \frac{\pi}{2}(r_2^4 - r_1^4) = \frac{\pi}{32}(d_2^4 - d_1^4)$$

The Torsion

The quantity GI_P/L, called the torsional stiffness of the bar, is the torque required to produce a unit angle of rotation. The torsional flexibility is the reciprocal of the stiffness, or L/GIP, and is defined as the angle of rotation produced by a unit torque. Thus, we have the following expressions:

$$k_T = \frac{GI_P}{L} \qquad f_T = \frac{L}{GI_P}$$

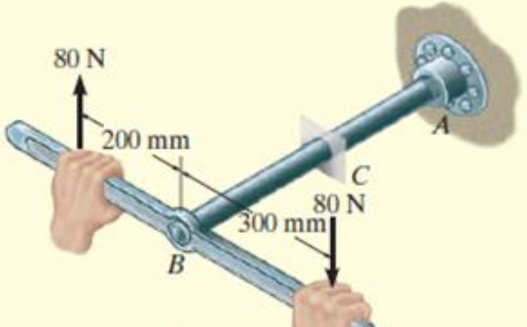
The pipe shown in figure has an inner diameter of 80 mm and an outer diameter of 100 mm. If its end is tightened against the support at *A* using a torque wrench at *B*, determine the shear stress developed in the material at the inner and outer walls along the central portion of the pipe when the 80-N forces are applied to the wrench.

SOLUTION

Internal Torque. A section is taken at an intermediate location *C* along the pipe's axis, Fig. 5–12*b*. The only unknown at the section is the internal torque **T**. We require

$$\Sigma M_y = 0; \quad 80\text{ N }(0.3\text{ m}) + 80\text{ N }(0.2\text{ m}) - T = 0$$

$$T = 40\text{ N}\cdot\text{m}$$



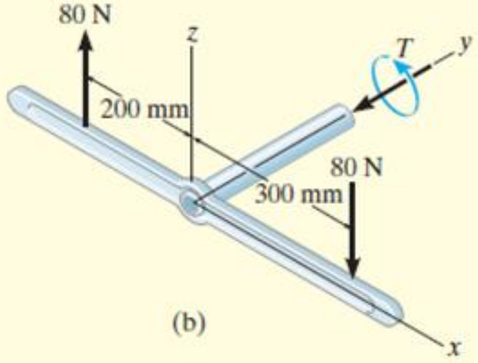
(a)

Section Property. The polar moment of inertia for the pipe's cross-sectional area is

$$J = \frac{\pi}{2}[(0.05\text{ m})^4 - (0.04\text{ m})^4] = 5.796(10^{-6})\text{ m}^4$$

Shear Stress. For any point lying on the outside surface of the pipe, $\rho = c_o = 0.05$ m, we have

$$\tau_o = \frac{Tc_o}{J} = \frac{40\text{ N}\cdot\text{m }(0.05\text{ m})}{5.796(10^{-6})\text{ m}^4} = 0.345\text{ MPa} \qquad \textit{Ans.}$$



(b)

And for any point located on the inside surface, $\rho = c_i = 0.04$ m, so that

$$\tau_i = \frac{Tc_i}{J} = \frac{40\text{ N}\cdot\text{m }(0.04\text{ m})}{5.796(10^{-6})\text{ m}^4} = 0.276\text{ MPa} \qquad \textit{Ans.}$$

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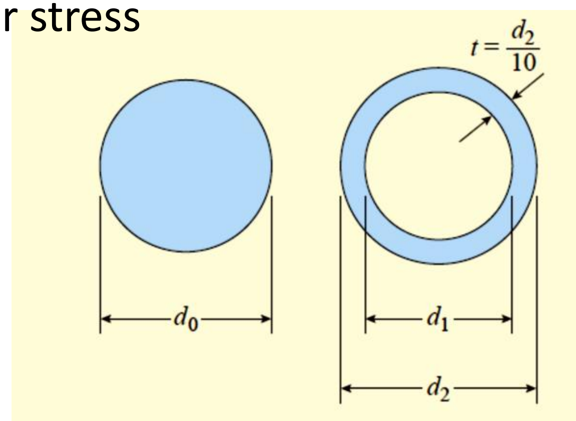
A steel shaft is to be manufactured either as a solid circular bar or as a circular tube. The shaft is required to transmit a torque of 1200 Nm without exceeding an allowable shear stress of 40 MPa nor an allowable rate of twist of 0.75^{O}/m. (Shear modulus of elasticity of steel is 78 GPa)

(a) Determine the required diameter d_O of the solid shaft. (b) Determine the required outer diameter d_2 of the hollow shaft if the thickness t of the shaft is specified as one-tenth of the outer diameter. (c) Determine the ratio of diameters (that is, the ratio d_2/d_O) and the ratio of weights of the hollow and solid shafts.

Solid shaft: The required diameter d_0 is determined either from the allowable shear stress



$$d_0^3 = \frac{16T}{\pi \tau_{allow}} = \frac{16(1200\ \mathrm{N\cdot m})}{\pi(40\ \mathrm{MPa})} = 152.8 \times 10^{-6}\ \mathrm{m}^3$$

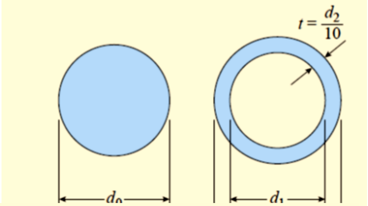
$$d_0 = 0.0535\ \mathrm{m} = 53.5\ \mathrm{mm}$$

In the case of the allowable rate of twist, we start by finding the required polar moment of inertia

$$I_P = \frac{T}{G\theta_{allow}} = \frac{1200\ \mathrm{N\cdot m}}{(78\ \mathrm{GPa})(0.75°/\mathrm{m})(\pi\ \mathrm{rad}/180°)} = 1175 \times 10^{-9}\ \mathrm{m}^4$$

$$d_0^4 = \frac{32 I_P}{\pi} = \frac{32(1175 \times 10^{-9}\ \mathrm{m}^4)}{\pi} = 11.97 \times 10^{-6}\ \mathrm{m}^4$$

$$d_0 = 0.0588\ \mathrm{m} = 58.8\ \mathrm{mm}$$



Hollow shaft: The required diameter is either the allowable shear stress or the allowable rate of twist. We begin by noting that the outer diameter of the bar is d_2 and the inner diameter is

$$d_1 = d_2 - 2t = d_2 - 2(0.1d_2) = 0.8d_2$$

The polar moment of inertia

$$I_P = \frac{\pi}{32}(d_2^4 - d_1^4) = \frac{\pi}{32}\left[d_2^4 - (0.8d_2)^4\right] = \frac{\pi}{32}(0.5904d_2^4) = 0.05796d_2^4$$

$$\tau_{\text{allow}} = \frac{Tr}{I_P} = \frac{T(d_2/2)}{0.05796\,d_2^4} = \frac{T}{0.1159d_2^3}$$

Rearranging, we get

$$d_2^3 = \frac{T}{0.1159\tau_{\text{allow}}} = \frac{1200\text{ N·m}}{0.1159(40\text{ MPa})} = 258.8 \times 10^{-6}\text{ m}^3$$

Solving for d_2 gives

$$d_2 = 0.0637\text{ m} = 63.7\text{ mm}$$

In the case of the allowable rate of twist,

$$\theta_{\text{allow}} = \frac{T}{G(0.05796d_2^4)}$$

$$d_2^4 = \frac{T}{0.05796G\theta_{\text{allow}}}$$

$$= \frac{1200\text{ N·m}}{0.05796(78\text{ GPa})(0.75°/\text{m})(\pi\text{ rad}/180°)} = 20.28 \times 10^{-6}\text{ m}^4$$

Solving for d_2 gives

$$d_2 = 0.0671\text{ m} = 67.1\text{ mm}$$

Ratios of diameters and weights. The ratio of the outer diameter of the hollow shaft to the diameter of the solid shaft

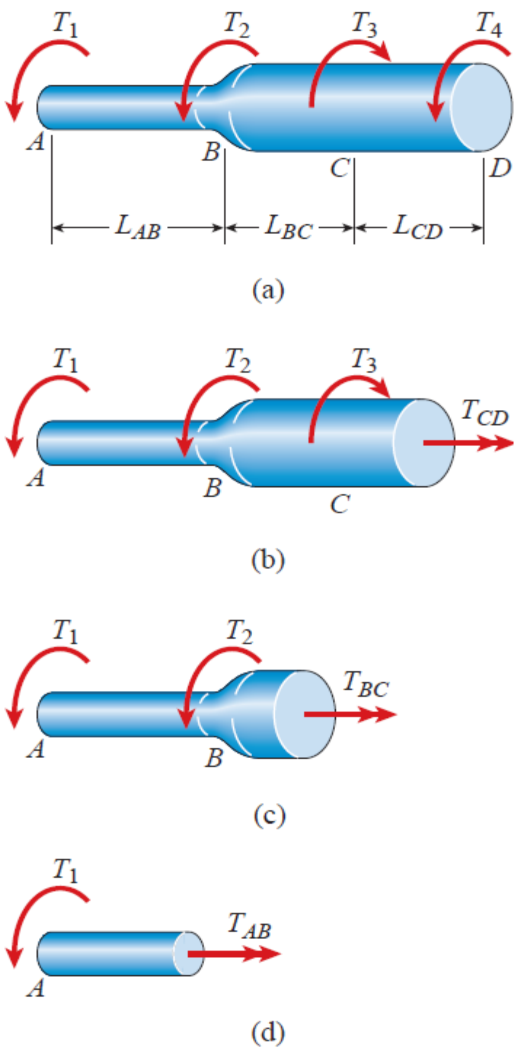
$$\frac{d_2}{d_0} = \frac{67.1\text{ mm}}{58.8\text{ mm}} = 1.14$$

the weights of the shafts are proportional to their cross-sectional areas

$$\frac{W_{\text{hollow}}}{W_{\text{solid}}} = \frac{A_{\text{hollow}}}{A_{\text{solid}}} = \frac{\pi(d_2^2 - d_1^2)/4}{\pi d_0^2/4} = \frac{d_2^2 - d_1^2}{d_0^2}$$

$$= \frac{(67.1\text{ mm})^2 - (53.7\text{ mm})^2}{(58.8\text{ mm})^2} = 0.47$$

Non-uniform Torsion



Case 1: Bar consisting of prismatic segments with constant torque throughout each segment

T_{CD} = - T_1 - T_2 + T_3

T_{BC} = - T_1 - T_2

T_{AB} = - T_1

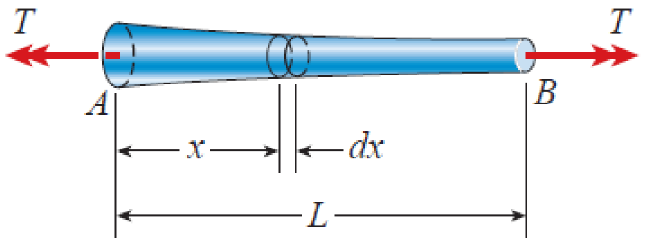
The total angle of twist of one end of the bar with respect to the other is then obtained by algebraic summation, as follows

$$\phi = \phi_1 + \phi_2 + \ldots + \phi_n$$

$$\boxed{\phi = \sum_{i=1}^{n} \phi_i = \sum_{i=1}^{n} \frac{T_i L_i}{G_i (I_P)_i}}$$

Non-uniform Torsion

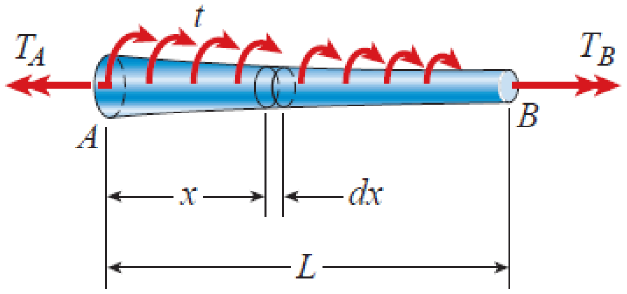
Case 2: Bar with continuously varying cross sections and constant torque



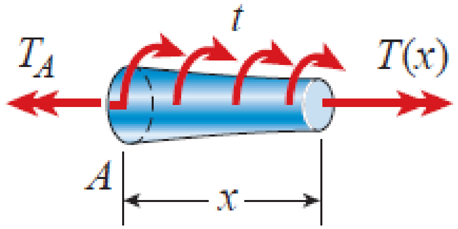
$$d\phi = \frac{T\,dx}{GI_P(x)}$$

$$\phi = \int_0^L d\phi = \int_0^L \frac{T\,dx}{GI_P(x)}$$

Case 3: Bar with continuously varying cross sections and continuously varying torque



(a)



$$\phi = \int_0^L d\phi = \int_0^L \frac{T(x)\,dx}{GI_P(x)}$$

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Power Transmission

Shafts and tubes having circular cross sections are often used to transmit power developed by a machine. They are subjected to a torque that depends on the power generated by the machine and the angular speed of the shaft.

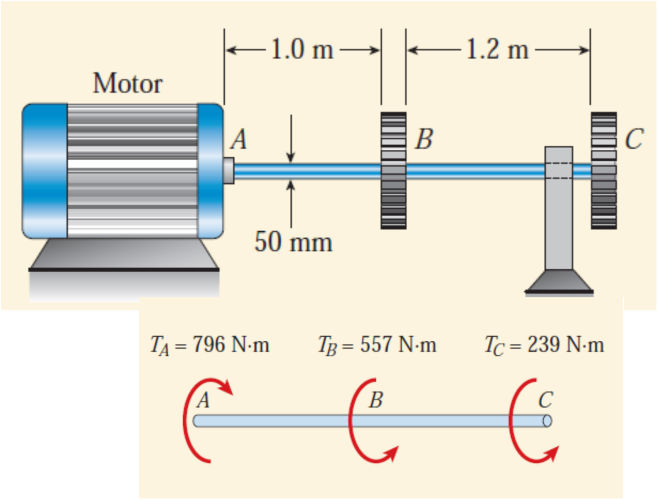
$$P = T\omega = T\frac{d\theta}{dt}$$

Power is defined as the work performed per unit of time. In the SI system, power is in watts when torque is measured in newton-meters per second [1W=1 Nm/s]

For machinery, the frequency of a shaft's rotation, N, is often used. (Number of revolutions or cycles the shaft per minute). Then

$$P = 2\pi NT/60$$

A solid steel shaft ABC of 50 mm diameter is driven at A by a motor that transmits 50 kW to the shaft at 10 Hz. The gears at B and C drive machinery requiring power equal to 35 kW and 15 kW, respectively. Compute the maximum shear stress τ_{max} in the shaft and the angle of twist ϕ_{AC} between the motor at A and the gear at C. (Use G = 80 Ga)



The torques applied to the shaft by the motor and the two gears are

$$T_A = \frac{P}{2\pi f} = \frac{50 \text{ kW}}{2\pi(10 \text{ Hz})} = 796 \text{ N·m}$$

$$T_B = \frac{P}{2\pi f} = \frac{35 \text{ kW}}{2\pi(10 \text{ Hz})} = 557 \text{ N·m}$$

$$T_C = \frac{P}{2\pi f} = \frac{15 \text{ kW}}{2\pi(10 \text{ Hz})} = 239 \text{ N·m}$$

The internal torques in the two segments of the shaft are now found by free body diagram $T_{AB} = 796$ N·m $\quad T_{BC} = 239$ N·m

The shear stress and angle of twist in segments are

$$\tau_{AB} = \frac{16 T_{AB}}{\pi d^3} = \frac{16(796 \text{ N·m})}{\pi(50 \text{ mm})^3} = 32.4 \text{ MPa}$$

$$\phi_{AB} = \frac{T_{AB} L_{AB}}{G I_P} = \frac{(796 \text{ N·m})(1.0 \text{ m})}{(80 \text{ GPa})\left(\frac{\pi}{32}\right)(50 \text{ mm})^4} = 0.0162 \text{ rad}$$

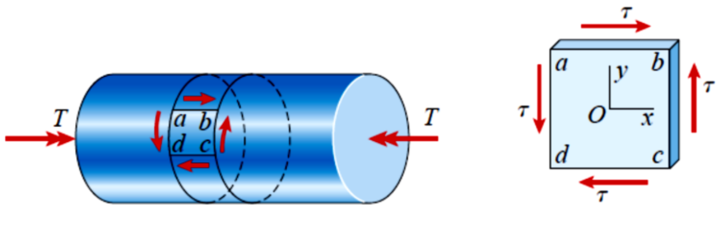
$$\tau_{BC} = \frac{16 T_{BC}}{\pi d^3} = \frac{16(239 \text{ N·m})}{\pi(50 \text{ mm})^3} = 9.7 \text{ MPa}$$

$$\phi_{BC} = \frac{T_{BC} L_{BC}}{G I_P} = \frac{(239 \text{ N·m})(1.2 \text{ m})}{(80 \text{ GPa})\left(\frac{\pi}{32}\right)(50 \text{ mm})^4} = 0.0058 \text{ rad}$$

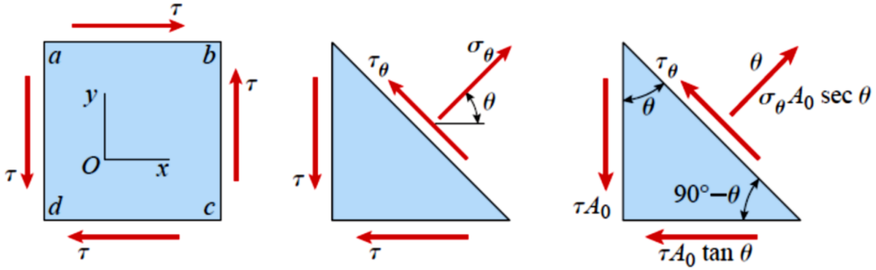
Maximum shear stress τmax = 32.4 MPa
Total Angle of twist = 0.0162+0.0058
= 0.0220 Rad
= 1.26°

Pure Shear Stresses and Strains

When a circular bar, either solid or hollow, is subjected to torsion, shear stresses act over the cross sections and on longitudinal planes.

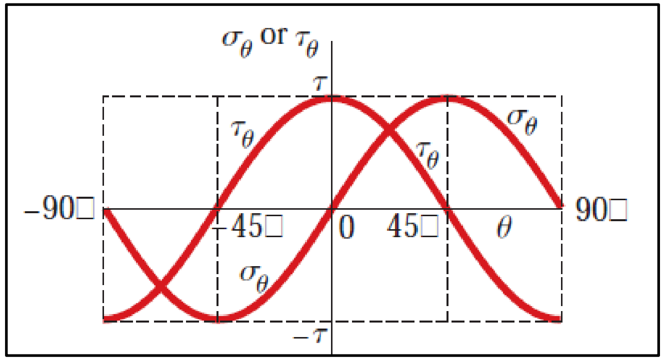


(a) (b)



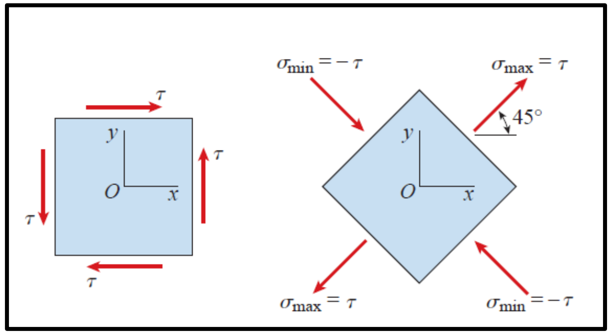
(a) (b) (c)

The horizontal and vertical faces of the triangular element have positive shear stresses τ, and the front and rear faces of the element are free of stress. The stresses σ_θ and τ_θ may now be determined from the equilibrium of the triangular element



$$\sigma_\theta = \tau \sin 2\theta \qquad \tau_\theta = \tau \cos 2\theta$$

If a stress element is oriented at an angle of 45°, both normal and shear stresses will be $\sigma_\theta = \tau_\theta$ and on other plane $\sigma_\theta = -\tau_\theta$



Strains in Pure Shear

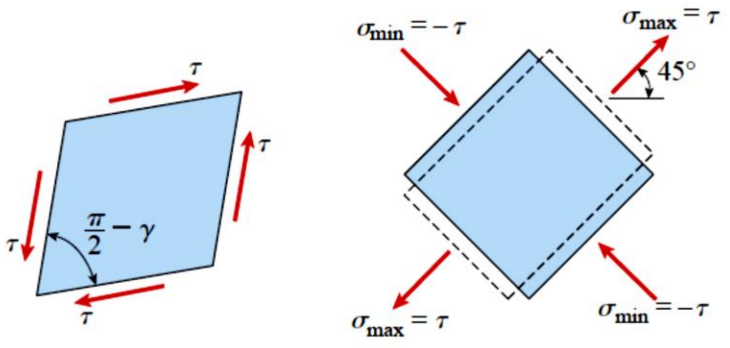
Consider the strains that exist in an element in pure shear. The shear strain γ is the change in angle between two lines that were originally perpendicular to each other. The lengths of the sides of the element, do not change when these shear deformations occur. This change in shape is called a shear distortion.

If the material is linearly elastic, the shear strain for the element oriented at θ=0 is related to the shear stress by Hooke's law in shear **$\gamma = \tau/G$**

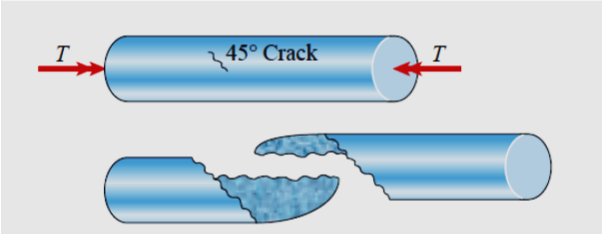
$$\epsilon_{max} = \frac{\tau}{E} + \frac{\nu\tau}{E} = \frac{\tau}{E}(1 + \nu)$$

The geometry of the deformed element to relate the shear strain γ to the normal strain ε_{max} in the 45° direction

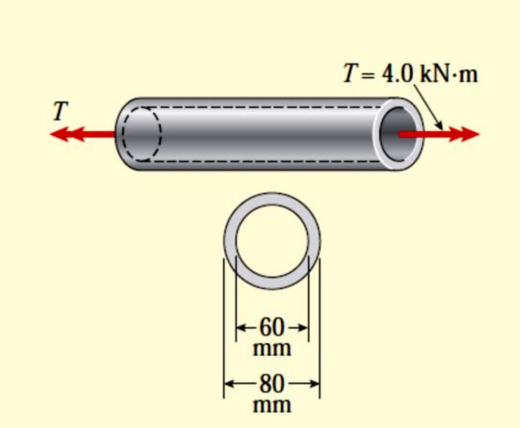
$$\boxed{\epsilon_{max} = \frac{\gamma}{2}}$$



(a) (b)



A circular tube with an outside diameter of 80 mm and an inside diameter of 60 mm is subjected to a torque T= 4.0 kNm. The tube is made of aluminum alloy 7075-T6 with G=27 GPa.
(a) Determine the maximum shear, tensile, and compressive stresses in the tube
(b) Determine the corresponding maximum strains in the tube
Show these stresses strains on properly oriented stress elements.



The maximum values of all three stresses (shear, tensile, & compressive) are

$$\tau_{\max} = \frac{Tr}{I_P} = \frac{(4000\ \mathrm{N\cdot m})(0.040\ \mathrm{m})}{\frac{\pi}{32}\left[(0.080\ \mathrm{m})^4 - (0.060\ \mathrm{m})^4\right]} = 58.2\ \mathrm{MPa}$$

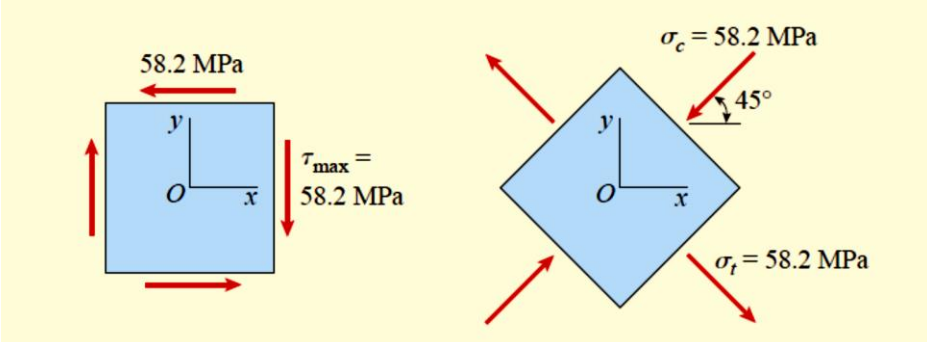
$$\sigma_t = 58.2\ \mathrm{MPa} \qquad \sigma_c = -58.2\ \mathrm{MPa}$$

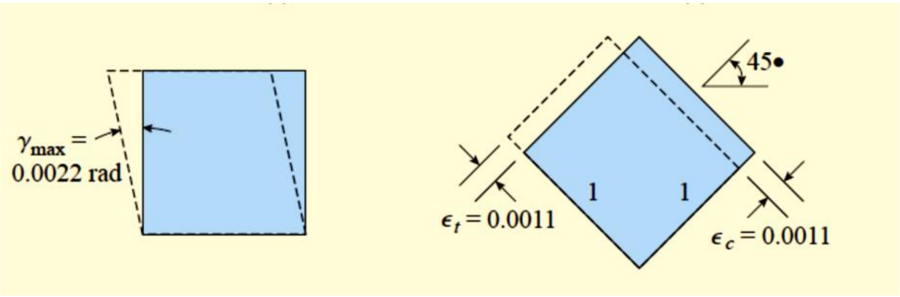
The maximum shear strain in the tube is obtained from

$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{58.2\ \mathrm{MPa}}{27\ \mathrm{GPa}} = 0.0022\ \mathrm{rad}$$

$$\epsilon_{\max} = \frac{\gamma_{\max}}{2} = 0.0011$$

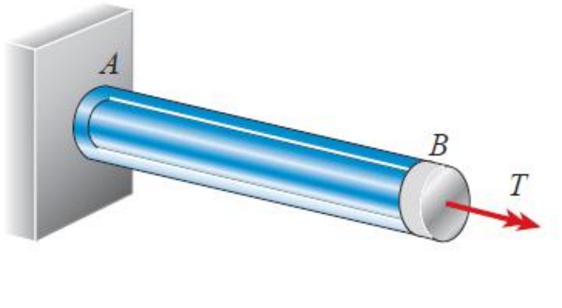
$$\epsilon_t = 0.0011 \qquad \epsilon_c = -0.0011$$



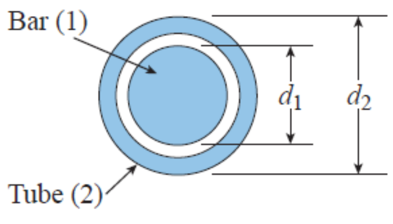


Statically Indeterminate Torsional Members

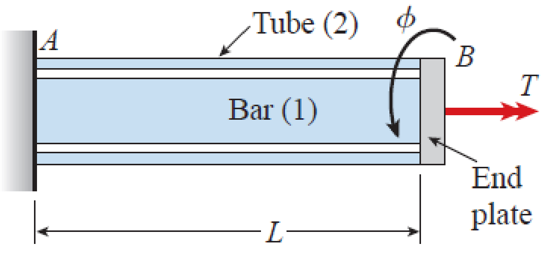
In statically indeterminate problems all internal torques and all reactions can be obtained from free-body diagrams, equations of equilibrium and additional restraints, such as fixed supports.



(a)



(b)



(c)

- The first step in the analysis is to write the equations of equilibrium (torques)

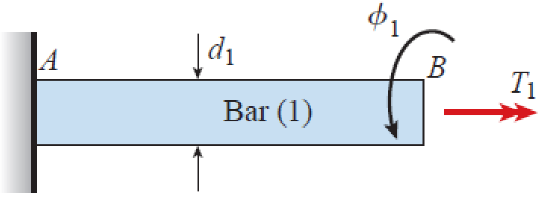
$$T_1 + T_2 = T$$

- The second step in the analysis is to formulate equations of compatibility, based upon physical conditions pertaining to the (angles of twist)

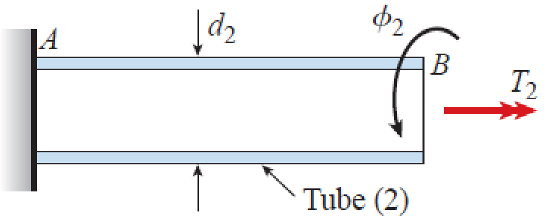
$$\phi_1 = \phi_2$$

- The third step is to relate the angles of twist to the torques by (torque-displacement relations), such as TL/GIP.

$$\phi_1 = \frac{T_1 L_1}{G_1 I_{P1}} \qquad \phi_2 = \frac{T_2 L_2}{G_2 I_{P2}}$$



(d)



(e)

THANK YOU